TENSOR PRODUCTS AND SUCCESSIVE APPROXIMATIONS FOR PARTIAL DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

A method for successive approximations for partial differential equations is presented. The limiting function in the process depends on the initial value. Convergence is proved by means of some identities and inequalities involving tensor products between multi-linear functions on a vector space.

1. Introduction. In this paper there is presented a method of successive approximations for partial differential equations. Although the method yields iterants for wide class of problems, it is studied here in connection with constant coefficient linear equations. To get a rough indication of the method, consider a real-valued function U which for some positive integer n and some positive integer m is n-times continuously differentiable in some convex domain D of E_m which contains the origin. Using Fréchet derivatives one has (c.f. [2], p. 92) the Taylor formula

$$U(x) = \sum_{q=0}^{n-1} (1/q!)U^{(q)}(0)x^q + \int_0^1 dj[(1-j)^{n-1}/(n-1)!]U^{(n)}(jx)x^n$$

for all x in D. Now, suppose one wants to solve a certain nth order partial differential equation (*) on a subset of D. As a first step one might attempt to find a real-valued function TU on a subset of D which is "closer" to a solution of (*) than is U by taking, for each x in D,

$$(TU)(x) = \sum_{q=0}^{n-1} (1/q!)U^{(q)}(0)x^q + \int_0^1 dj[(1-j)^{n-1}/(n-1)!]\{S_{jx}[U^{(n)}(jx)]\}x^n$$

where, if w is in D, $S_w[U^{(n)}(w)]$ is an appropriate "minimal modification" of $U^{n)}$ (w) which satisfies (*) at w.

This term "minimal modification" is made precise and convergence of T^qU to a solution of (*) as $q \to \infty$ is proved for analytic U for the case in which (*) is a

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constant coefficient linear n-th order partial differential equation with no lower order terms and with polynomial inhomogeneous part. It is noted that for any n-times continuously differentiable function U on D, $TU = U$ if and only if U is already a solution to (*).

The limiting function for this successive approximation process is *not* (unlike the case of successive approximation for ordinary differential equations) independent of the "initial estimate". In fact the transformation which pairs initial estimates with corresponding limiting functions is a projection in a space of analytic functions.

The convergence argument depends on some inequalities and identities concerning symmetric products between symmetric multilinear functions. Some preliminaries concerning tensor products are presented in Section 2 after certain inner product spaces of multilinear functions are defined. Section 3 gives some theorems on transformations in spaces of multilinear functions. Section 4 gives the main result and in Section 5 some examples are considered.

2. This section contains some preliminaries concerning multilinear functions. For this entire section, suppose that m is a positive integer. If n is a positive integer, denote by $M_{m,n}$ the linear space of real n-linear functions on the real, m dimensional inner product space E_m and denote by $S_{m,n}$ the subspace of symmetric n-linear functions. As in ([1], p. 206), for example, if each of n and k is a positive integer, u is in $M_{m,n}$, w is in $M_{m,k}$, then the tensor product $u \otimes w$ is the member of $M_{m,n+k}$ such that $(u \otimes w)(x_1, \dots, x_{n+k}) = u(x_1, \dots, x_n) w(x_{n+1}, \dots, x_{n+k})$ for all $x_1, \dots, x_{n+k}$ in E_m. If $n = 1$ and u is in E_m, then $u \otimes w = u' \otimes w$ and $w \otimes u = w \otimes u'$ when u' is dual to u. Similarly, if each of u and w is in E_m, then $u \otimes w = u' \otimes w'$ when u', w' are dual to u, w respectively. The tensor product so defined gives a bilinear function from $M_{m,n} \times M_{m,k}$ to $M_{m,n+k}$. In addition, $M_{m,n}$ is generated linearly by elements of the form $x_1 \otimes \dots \otimes x_n$ such that each of $x_1, \dots, x_n$ is in E_m ([1], p. 206).

As noted in ([1], p. 79), an inner product in $M_{m,n}$ is introduced by specifying that

$$(x_1 \otimes \dots \otimes x_n, y_1 \otimes \dots \otimes y_n) = (x_1, y_1) \dots (x_n, y_n)$$

if each of $x_1, \dots, x_n$, $y_1, \dots, y_n$ is in E_m. If each of u and w is in $M_{m,n}$, the inner product of u and w is denoted simply by uw. The same convention will be used for the inner product of a vector x in E_m with a vector y in E_m, that is, the inner product of x with y will be denoted by xy.

If t is a positive integer greater than n, w is in $M_{m,t}$ and each of $x_1, \dots, x_{t-n}$ is in E_m, then $w(x_1, \dots, x_{t-n})$ denotes the element g of $M_{m,n}$ such that if each of $y_1, \dots, y_n$ is in E_m, then $g(y_1, \dots, y_n) = w(y_1, \dots, y_n, x_1, \dots, x_{t-n})$, i.e., $(w(x_1, \dots, x_{t-n}))(y_1, \dots, y_n) = w(y_1, \dots, y_n, x_1, \dots, x_{t-n})$. If, in addition, z is in $M_{m,n}$, then the inner product $wz = zw$ is defined as the member h of $M_{m,n-t}$ so that $h(x_1, \dots, x_{t-n}) = z(w(x_1, \dots, x_{t-n}))$ if each of $x_1, \dots, x_{t-n}$ is in E_m.

If each of k and n is a positive integer, denote by $J_{k,n}$ the set of all sequences $\{p_i\}_{i=1}^n$ such that p_i is in $\{1,..,k\}$, $i=1,..,n$, and denote by $K_{k,n}$ the set of all sequences $\{q_i\}_{i=1}^k$ such that q_i is in $\{0,1,..,n\}$, $i=1,..,k$, and $\Sigma_{i=1}^k q_i = n$. Denote by $\alpha_{k,n}$ the function from $J_{k,n}$ to $K_{k,n}$ such that if $p=\{p_i\}_{i=1}^n$ is in $J_{k,n}$, then $\alpha_{k,n}(p)$ is the element $\{q_i\}_{i=1}^k$ of $K_{k,n}$ such that if i is in $\{1,..,k\}$, then q_i is the number of integers j in $\{1,..,n\}$ such that $p_j = i$.

If each of t,s and r is a positive integer, $r \leqq s$, z is in $M_{m,t}$, $x = x_1,..,x_s$ is a sequence each term of which is in E_m and $p=\{p_i\}_{i=1}^r$ is a member of $J_{s,r}$, then $z(x_p)$ denotes $z(x_{p_1},..,x_{p_r})$.

By convention, if z is in $M_{m,n}$ and x is in E_m, then $zx = zx'$ where x' is dual to x and hence, if each of $y_1,..,y_{n-1}$ is in E_m, $(zx)(y_1,..,y_{n-1}) = (z(y_1,..,y_{n-1}))x' = z(x,y_1,..,y_{n-1})$ whereas $(z(x))(y_1,..,y_{n-1}) = z(y_1,..,y_{n-1},x)$ so that $zx = z(x)$ need not hold. Sometimes zx is written as $(z)(x)$.

Suppose that each of n and k is a positive integer, z is in $M_{m,n+k}$, w is in $M_{m,n}$, u is in $M_{m,k}$ and $x = x_1,..,x_m$ is an orthonormal sequence in E_m. Then

$$(zw)u = \sum_{p\in J_{m,k}} ((zw)(x_p))u(x_p) = \sum_{p\in J_{m,k}} ((z(x_p))w)u(x_p)$$
$$= \sum_{p\in J_{m,k}} \sum_{q\in J_{m,n}} (z(x_p))(x_q)w(x_q)u(x_p).$$

Hence if $h = w\otimes u$,

$$(zw)u = \sum_{p\in J_{m,k}} \sum_{q\in J_{m,n}} (z(x_p))(x_q)(h(x_p))(x_q)$$
$$= \sum_{r\in J_{m,n+k}} z(x_r)h(x_r) = zh.$$

Hence $w\otimes u$ is the (necessarily unique) element h of $M_{m,n+k}$ soch that $zh = (zw)u$ for all z in $M_{m,n+k}$.

If each of n and k is a positive integer, each of w and u is in $M_{m,n}$ and each of x and y is in $M_{m,k}$, then $(w\otimes x)(u\otimes y) = (wu)(xy)$. This is a simple consequence of the defining equation for inner product in $M_{m,n+k}$. From this one has that $\| w\otimes x\|^2 = (w\otimes x)(w\otimes x) = (ww)(xx) = \|w\|^2\|x\|^2$ so that $\|w\otimes x\| = \|w\|\|x\|$.

If u is in $M_{m,n}$ and B_n denotes the set of all permutations on $\{1,..,n\}$, then Su (c.f. [1], p. 190) defined by

$$((Su)(x_1,..,x_n) = (1/n!) \sum_{\sigma\in B_n} u(x_{\sigma(1)},..,x_{\sigma(n)})$$

for all $x_1,..,x_n$ in E_m is called the symmetric part of u. S itself is called the symmetrizer operator.

Now suppose that v is in $M_{m,n}$ and

$$Lu = \sum_{p\in J_{m,n}} u(x_p)v(x_p) = uv$$

for all u in $S_{m,n}$.

If $q = \{q_i\}_{i=1}^m$ is in $K_{m,n}$, then $u(x^q)$ denotes $u(x_1^{q_1}, \ldots, x_m^{q_m})$. Hence

$$Lu = \sum_{q \in K_{m,n}} \sum_{t \in \alpha_{m,n}^{-1}(q)} u(x_t)v(x_t) = \sum_{q \in K_{m,n}} u(x^q) \sum_{t \in \alpha_{m,n}^{-1}(q)} v(x_t)$$

$$= \sum_{p \in J_{m,n}} u(x_p)\bar{v}(x_p) = u\bar{v}$$

where $\bar{v}(x_p) = |\alpha^{-1}(\alpha(p))|^{-1} \sum_{t \in \alpha^{-1}(\alpha(p))} v(x_t)$ for all p in $J_{m,n}$. A simple counting argument gives that $\bar{v} = Sv$. Hence, if v is in $M_{m,n}$, then Sv is the (necessarily unique) element w of $S_{m,n}$ such that $uv = uw$ for all u in $S_{m,n}$. This is equivalent to saying that S is the orthogonal projection of $M_{m,n}$ onto $S_{m,n}$, i.e., $S^2 = S$, $S^* = S$.

Suppose that u is in $M_{m,n}$, w is in $M_{m,k}$. Denote $S(u \otimes w)$ by $u \cdot w$. This is, except for the numerical factor $(k-n)!/k!n!$ the symmetric product of u and w in ([1], p. 220). Hence it is concluded that commutivity, associativity and distributivity with respect to addition in the spaces of multilinear functions hold. In addition observe that if r is in $S_{m,n+k}$, then $r(u \cdot w) = (Sr)(u \cdot w) = r(S(u \cdot w)) = r(S(u \otimes w)) = (Sr)(u \otimes w) = r(u \otimes w) = (ru)w$ so that one has the useful fact that $r(u \cdot w) = (ru)w$. Note also that $\| u \cdot w \| = \| S(u \otimes w) \| \leq \| u \otimes w \| = \| u \| \| w \|$.

Suppose now that each of a, b and c is a positive integer, $c \geq a + b$, z is in $S_{m,c}$, u is in $S_{m,a}$ and w is in $S_{m,b}$. To see that $(zu)w = (zw)u$, consider the following: If $c = a + b$, the result follows from the fact that $u \cdot w = w \cdot u$. Suppose that $c > a + b$ and denote $c - (a + b)$ by n. Denote by q a member of $S_{m,n}$. Then $z((q \cdot u) \cdot w) = (zw)(q \cdot u) = ((zw)u)q$ and likewise $z((q \cdot w) \cdot u) = ((zu)w)q$. Hence $((zu)w)q = ((zw)q = ((zw)u)q$ for all q in $S_{m,n}$ and so $(zw)u = (zu)w$.

From the definition of tensor product it follows that if y is in E_m and n is a positive integer, then $(\underbrace{y \otimes \cdots \otimes y}_{n})\ (x_1, \ldots, x_n) = (yx_1) \cdots (yx_n) = (yx_{\sigma(1)}) \cdots (yx_{\sigma(n)})$ for any permutation σ on $\{1, \ldots, n\}$. Hence $\underbrace{y \otimes \cdots \otimes y}_{n}$ is symmetric and so is equal to $\underbrace{y \cdot \ldots \cdot y}_{n}$ and will be denoted by y^n.

From ([2], p. 92) it follows that if each of u and w is in $S_{m,n}$ and $uy^n = wy^n$ for all y in E_m, then $u = w$. Hence if u is in $S_{m,n}$, v is in $S_{m,k}$ and $u \neq 0 \neq v$ it follows that $u \cdot v \neq 0$. To see this, suppose otherwise. Then the real valued functions f and g on E_m such that $f(y) = uy^n$ and $g(y) = vy^k$ for all Y in E_m have the property that $f(y)g(y) = (uy^n)(vy^k) = (u \cdot v)y^{n+k} = 0$ for all y in E_m and hence either f or g must be zero on some open subset of E_m. But from this it would follow that either f or g is identically zero on E_m, i.e., $u = 0$ or $v = 0$, a contradiction.

Suppose that each of n and k is a positive integer, A is a non-zero member of $S_{m,n}$ and M is the transformation from $S_{m,k}$ to $S_{m,k}$ such that $Mu = A(A \cdot u)$ for all u in $S_{m,k}$. Then M is linear, symmetric and (strictly) positive. The linearity of M is clear enough. To see that M is positive and symmetric, suppose that each of u and

v is in $S_{m,k}$. Then $(Mu)u = (A(A \cdot u))u = (A \cdot u)(A \cdot u) = \| A \cdot u \|^2 > 0$ unless $u = 0$. Also, $(Mu)v = (A(A \cdot u))v = (A \cdot u)(A \cdot v) = u(A(A \cdot v)) = u(Mv)$.

3. Suppose that each of k, m and n is a positive integer and A is a member of $S_{m,n}$ such that $\| A \| = 1$. Denote by M the transformation on $S_{m,k}$ such that $Mu = A(A \cdot u)$ for all u in $S_{m,k}$. From section 2, M is invertible, symmetric and positive. Hence M^{-1} has these properties. Denote by P the transformation on $S_{m,n+k}$ so that $Pw = A \cdot [M^{-1}(Aw)]$ for all w in $S_{m,n+k}$. It is easy to check that P is symmetric and idempotent and hence each of P and $I - P$ is an orthogonal projection.

Denote by G the transformation on $S_{m,n+k}$ such that $Gw = A \cdot (Aw)$ for all w in $S_{m,n+k}$. Note that if u is in $S_{m,k}$ and p is a positive integer, then $G(A \cdot u) = A \cdot (Mu)$, $(I - G)(A \cdot u) = A \cdot [(I - M)u]$, $G^p[A \cdot u] = A \cdot (M^p u)$ and $(I - G)^p(A \cdot u) = A \cdot [(I - M)^p u]$.

Denote $I - P$ by L and denote $I - G$ by Q. If t is a positive integer and Z is a linear transformation on $S_{m,t}$, then $|Z|$ denotes $\text{lub}_{x \in S_{m,t}, \|x\| = 1} \| Zx \|$.

Theorem. $QL = LQ = L$ and $\lim_{p \to \infty} Q^p = L$.

Proof. To show that $QL = LQ = L$ it is sufficient to show that $PG = GP = G$ since $QL = (I - G)(I - P) = I - G - P + GP$ and $LQ = I - G - P + PG$. If w is in S_{m+n+k}, then $PGw = A \cdot (M^{-1}(A(Gw))) = A \cdot (M^{-1}(A(A \cdot (Aw))))$ $= A \cdot (M^{-1}M(Aw)) = A \cdot (Aw) = Gw$ and $GPw = A \cdot (A(A \cdot (M^{-1}(Aw)))$ $= A \cdot (MM^{-1}(Aw)) = A \cdot (Aw) = Gw$. Hence $PG = GP = G$.

If w is in $S_{m,n+k}$ and $Aw = 0$, then $Qw = w - A \cdot (Aw) = w$ and $Lw = w - A \cdot (M^{-1}(Aw)) = w$.

If for some u in $S_{m,k}$, $w = A \cdot u$, then $Lw = A \cdot u - A \cdot (M^{-1}(A(A \cdot u)))$ $= A \cdot u - A \cdot u = 0$. Since if u is in $S_{m,k}$, $(Mu)u = \| A \cdot u \|^2 \leqq \| A \|^2 \| u \|^2 = \| u \|^2$, it follows that $|M| \leqq 1$. Inasmuch as M is symmetric and positive it must be that $|I - M| < 1$. From this it follows that if w is in $S_{m,n+k}$ and $w = A \cdot u$ for some u in $S_{m,k}$, then

$$\lim_{p \to \infty} \| Q^p w \| = \lim_{p \to \infty} \| A \cdot ((I - M)^p u) \| \leqq \lim_{p \to \infty} \| (I - M)^p u \| = 0.$$

Suppose w is in $S_{m,n+k}$. Then $\lim_{p \to \infty} Q^p(Pw) = 0$ since Pw is of the form $A \cdot u$ for some u in $S_{m,k}$. Hence,

$$Q^p w = Q^p(Pw) + Q^p((I - P)w) = Q^p(Pw) + Q^p(Lw) = Q^p(Pw) + Lw \to Lw \text{ as } p \to \infty.$$

The theorem follows since $S_{m,n+k}$ is finite dimensional.

Suppose now in addition to the above notation and definitions that R is in $S_{m,k}$ and each of Z and V is a transformation on $S_{m,n+k}$ such that $Zw = w + A \cdot [R - (Aw)] = (I - G)w + A \cdot R$, $Vw = w + A \cdot (M^{-1}(R - (Aw)))$ $= (I - P)w + A \cdot (M^{-1}R) = Lw + A \cdot (M^{-1}R)$ for all w in $S_{m,n+k}$.

THEOREM. *$VZ = ZV = V$ and $\lim_{p\to\infty} Z^p w = Vw$ for all w in $S_{m,n+k}$.*

Proof. If w is in $S_{m,n+k}$, then $ZVw = (I-G)Vw + A\cdot R = (I-G)[(I-P)w + A\cdot(M^{-1}R)] + A\cdot R = QLw + A\cdot(M^{-1}R) - A\cdot(A(A\cdot(M^{-1}R))) + A\cdot R = Lw + A\cdot(M^{-1}R) - A\cdot(MM^{-1}R) + A\cdot R = Lw + A\cdot(M^{-1}R) = Vw$ and $VZw = LZw + A\cdot(M^{-1}R) = L(Qw + A\cdot R) + A\cdot(M^{-1}R) = Lw + L(A\cdot R) + A\cdot(M^{-1}R) = Lw + A\cdot R - A\cdot(M^{-1}(A(A\cdot R))) + A\cdot(M^{-1}R) = Lw + A\cdot R - A\cdot(M^{-1}MR) + A\cdot(M^{-1}R) = Lw + A\cdot(M^{-1}R) = Vw$. Hence $ZV = VZ = V$. By induction, if p is a positive integer and w is in $S_{m,n+k}$, $Z^p w = Q^p w + [Q^{p-1}(A\cdot R) + \,..\, + Q(A\cdot R) + A\cdot R]$. Remember that if p is a positive integer, $Q^p(A\cdot R) = A\cdot((I-M)^p R)$ and hence $Z^p w = Q^p w + A\cdot([(I-M)^{p-1} + \,..\, + (I-M) + I]R)$. But it was seen in the proof of the preceding theorem that $|I-M| < 1$. Hence $\lim_{p\to\infty}[(I-M)^{p-1} + \,..\, + (I-M) + I] = M^{-1}$. Since $\lim_{p\to\infty} Q^p w = Lw$ it follows that $\lim_{p\to\infty} Z^p w = Lw + A\cdot(M^{-1}R) = Vw$.

4. Suppose that each of m, n and k is a positive integer, A is in $S_{m,n}$ and $\|A\| = 1$. Suppose in addition that D is a convex open subset of E_m which contains the origin. If p is a positive integer, then $C^p(D)$ denotes the set of all real-valued functions U with domain D such that U has a continuous pth Fréchet derivative (general reference for Fréchet derivatives is [2]). Denote by $A(D)$ those real-valued functions U on D such that if x is in D, $U(x) = \sum_{q=0}^{\infty}(1/q!)U^{(q)}(0)x^q$ and $\sum_{q=0}^{\infty}(1/q!)\|U^{(q)}(0)\|\,\|x\|^q$ converges. Denote by $P(D)$ those members of $A(D)$ which are polynomials. Denote by R a member of $C^0(D)$ and denote by T the transformation from $C^n(D)$ to $C^0(D)$ such that if U is in $C^n(D)$, then

$$(TU)(x) = \sum_{q=0}^{n-1}(1/q!)U^{(q)}(0)x^q + \int_0^1 dj[(1-j)^{n-1}/(n-1)!]\{U^{(n)}(jx) - [A(U^{(n)}(jx)) - R(jx)]A\}x^n \text{ for all } x \text{ in } D.$$

THEOREM. *If U is in $C^n(D)$, then $A(U^{(n)}(x)) = R(x)$ for all x in D if and only if $TU = U$.*

Indication of proof. Suppose that U is in $C^n(D)$. If $A(U^{(n)}(x)) = R(x)$ for all x in D, then an examination of the Taylor formula in the introduction gives immediately that $TU = U$. On the other hand, if $TU = U$, then by the same Taylor formula,

$$\begin{aligned} 0 &= U(x) - (TU)(x) \\ &= \int_0^1 dj[(1-j)^{n-1}/(n-1)!][A(U^{(n)}(jx)) - R(jx)](Ax^n) \\ &= \int_0^1 d(j\|x\|)[(\|x\| - j\|x\|)^{n-1}/(n-1)!] \\ &\qquad [A(U^{(n)}(j\|x\|r_x)) - R(j\|x\|r_x)](Ar_x^n) \\ &= \int_0^{\|x\|} dj[(\|x\| - j)^{n-1}/(n-1)!][A(U^{(n)}(jr_x)) - R(jr_x)]\quad [Ar_x^n] \end{aligned}$$

for all x in D where $r_x = 0$ if $x = 0$ and $r_x = (1/\|x\|)\,x$ if $x \neq 0$, for all x in D. But this implies that $[A(U^{(n)}(x)) - R(x)](Ax^n) = 0$ for all x in D since the above integral is an n-fold iterated integral. Therefore $A(U^{(n)}(x)) = R(x)$ for all x in D since Ax^n can not be zero for all x in any dense subset of D.

A further note on the definition of T: If z is an element of $S_{m,n}$, c is a number and $Az = c$ is not satisfied by z, then $z - (Az - c)A$ is the closest element y of $S_{m,n}$ to z with the property that $Ay = c$. This observation was crucial in the discovery of the following theorem, which is the main result of this paper.

THEOREM. *If R is in $P(D)$ and U is in $A(D)$, then $\{T^pU\}_{p=1}^{\infty}$ converges uniformly on closed and bounded subsets of D to a member Y of $A(D)$ such that $A(Y^{(n)}(x)) = R(x)$ for all x in D.*

Proof. Suppose that R is in $P(D)$ and U is in $A(D)$ and denote by t a positive integer such that $R(x) = \Sigma_{p=0}^{t}(1/p!)R^{(p)}(0)x^q$ for all x in D. Note that if k is a positive integer, then $U^{(k)}(x) = \Sigma_{p=k}^{\infty}(1/(p-k)!)U^{(p)}(0)x^{p-k}$ and that $A(U^{(n)}(x)) = \Sigma_{p=n}^{\infty}(1/(p-k)!)A(U^{(p)}(0)x^{p-n}) = \Sigma_{p=n}^{\infty}(1/(p-n)!)[A(U^{(p)}(0))]x^{p-n}$ for all x in D. Then, if x is in D, $(TU)(x)$ may be rewritten

$$\begin{aligned}(TU)(x) &= \sum_{p=0}^{n-1}(1/p!)U^{(p)}(0)x^p \\ &\quad + \int_0^1 dj\,[(1-j)^{n-1}/(n-1)!]\,\Bigg\{\sum_{p=n}^{\infty}(1/(p-n)!)U^{(p)}(0)[(jx)^{p-n}\cdot x^n] \\ &\quad - \Bigg[\sum_{p=n}^{\infty}(1/(p-n)!)A(U^{(p)}(0)(jx)^{p-n}) - \sum_{p=0}^{t}(1/p!)R^{(p)}(0)(jx)^p\Bigg](Ax^n)\Bigg\} \\ &= \sum_{p=0}^{n-1}(1/p!)U^{(p)}(0)x^p \\ &\quad + \sum_{p=n}^{\infty}\Bigg[\int_0^1 dj(1-j)^{n-1}j^{p-n}/((n-1)!(p-n)!)\Bigg]U^{(p)}(0)x^p \\ &\quad - \sum_{p=n}^{\infty}\Bigg[\int_0^1 dj(1-j)^{n-1}j^{p-n}/((n-1)!(p-n)!)\Bigg][A(U^{(p)}(0)x^{p-n})](Ax^n) \\ &\quad + \sum_{p=0}^{t}\Bigg[\int_0^1 dj(1-j)^{n-1}j^p/((n-1)!p!)\Bigg](R^{(p)}(0)x^p)(Ax^n)\end{aligned}$$

$$= \sum_{p=0}^{n-1} (1/p!)U^{(p)}(0)x^p + \sum_{p=n}^{\infty} (1/p!)U^{(p)}(0)x^p$$

$$- \sum_{p=n}^{\infty} (1/p!)[A(U^{(p)}(0)x^{p-n})](Ax^n)$$

$$+ \sum_{p=0}^{t} (1/(p+n)!)(R^{(p)}(0)x^p)(Ax^n)$$

$$= \sum_{p=0}^{n-1} (1/p!)U^{(p)}(0)x^p$$

$$+ \sum_{p=n}^{n+t} (1/p!)[U^{(p)}(0) - A\cdot(A(U^{(p)}(0))) + A\cdot(R^{(p-n)}(0)]x^p$$

$$+ \sum_{p=n+t+1}^{\infty} (1/p!)[U^{(p)}(0) - A\cdot(A(U^{(p)}(0)))]x^p$$

$$= \sum_{p=0}^{n-1} (1/p!)U^{(p)}(0)x^p + \sum_{p=n}^{n+t} (1/p!)(Z_pU^{(p)}(0))x^p$$

$$+ \sum_{p=n+t+1}^{\infty} (1/p!)(Q_pU^{(p)}(0))x^p$$

where $Z_pw = w - A\cdot(Aw) + A\cdot R^{(p-n)}(0)$ for all w in $S_{m,p}$, $p = n,..,n+t$ and $Q_pw = w - A\cdot(Aw)$ for all w in $S_{m,p}$, $p = n+t+1$, $n+t+2,\dots$ Denote $A(A\cdot w)$ by M_pw for all w in $S_{m,p}$, $p = 0,1,..,$ denote $w - A\cdot(M_{p-n}^{-1}(Aw))$ by L_pw for all w in $S_{m,p}$, $p = n$, $n+1,..$ and denote $L_pw + A\cdot(M_{p-n}^{-1}(R^{(p-n)}(0)))$ by V_pw for all w in $S_{m,p}$, $p = n$, $n+1,\dots$

By induction, if q is a positive integer and x is in D,

$$(T^qU)(x) = \sum_{p=0}^{n-1} (1/p!)U^{(p)}(0)x + \sum_{p=n}^{n+t} (1/p!)(Z_p^qU^{(p)}(0))x^p$$

$$+ \sum_{p=n+t+1}^{\infty} (1/p!)(Q_p^qU^{(p)}(0))x^p.$$

The convergence of the above infinite series (and its membership in $A(D)$) is assured since $|Q_p| \leqq 1$ for each positive integer p and U is in $A(D)$.

By the theorems of the preceding section, if x is in D,

$$\lim_{q\to\infty} (T^qU)(x) = \sum_{p=0}^{n-1} (1/p!)U^{(p)}(0)x^p + \sum_{p=0}^{n+t} (1/p!)(V_pU^{(p)}(0))x^p$$

$$+ \sum_{p=n+t+1}^{\infty} (1/p!)(L_pU^{(p)}(0))x^p.$$

This limit is denoted by $(K_{A,R}U)(x)$ for all x in D. Membership of $K_{A,R}U$ in $A(D)$ is assured since $|L_p| \leqq 1$, $p = n+t$, $n+t+1,\dots$ That convergence of $\{T^qU\}_{q=1}^{\infty}$

to $K_{A,R}U$ is uniform on closed and bounded subsets of $A(D)$ may be seen from the inequality:

$$|(T^qU)(x)-(T^{q+r}U)(x)| \leqq \sum_{p=n}^{n+t}(1/p!)|(Z_p^qU^{(p)}(0))x^p-(Z_p^{q+r}U^{(p)}(0))x^p|$$

$$+\sum_{p=n+t+1}^{b}(1/p!)|(Q_p^qU^{(p)}(0))-(Q_p^{p+r}U^{(p)}(0))x^p|$$

$$+\ 2\sum_{p=b+1}^{\infty}(1/p!)\|U^{(p)}(0)\|\,\|x\|^p$$

for all x in D provided that each of q, r and b is a positive integer such that $b>n+t$.

Denote $K_{A,R}U$ by Y. Then $A(Y^{(n)}(x))=R(x)$ for all x in D since, if x is in D,

$$Y^{(n)}(x)=\sum_{p=n}^{n+t}(1/(p-n)!)[V_pU^{(p)}(0)]x^{p-n}+\sum_{p=n+t+1}^{\infty}(1/(p-n)![L_pU^{(p)}(0)]x^{p-n}$$

and hence

$$A(Y^{(n)}(x))=\sum_{p=n}^{n+t}(1/(p-n)!)A\{[V_pU^{(p)}(0)]x^{p-n}\}$$

$$+\sum_{p=n+t+1}^{\infty}(1/(p-n)!)A\{[L_pU^{(p)}(0)]x^{p-n}\}$$

$$=\sum_{p=n}^{n+t}(1/(p-n)!)(A(V_pU^{(p)}(0)))x^{p-n}$$

$$+\sum_{p=n+t+1}^{\infty}(1/(p-n)!)(A(L_pU^{(p)}(0)))\,x^{p-n}$$

$$=\sum_{p=n}^{n+t}(1/(p-n)!)R^{(p-n)}(0)x^{p-n}=\sum_{p=0}^{t}(1/p!)R^{(p)}(0)x^p=R(x)$$

since

$$A(L_pU^{(p)}(0))=A(U^{(p)}(0))-A(A\cdot(M_{p-n}^{-1}(A(U^{(p)}(0)))))$$
$$=A(U^{(p)}(0))-M_{p-n}M_{p-n}(A(U^{(p)}(0)))=0$$

and hence

$$A(V_pU^p\ (0))=A(L_pU^{(p)}(0))+A(A\cdot(M_{p-n}^{-1}(R^{(p-n)}(0))))$$
$$=M_{p-n}M_{p-n}^{-1}(R^{(p-n)}(0))=R^{(p-n)}(0),\ p=n,n+1,\ldots$$

This completes a proof of the theorem.

5. Examples. In this final section some examples are considered and some comments made concerning boundary value problems. Suppose that each of m and n is a positive integer, A is in $S_{m,n}$, $\|A\| = 1$ and D is a circular open disk with center the origin of E_m. Then if U is in $A(D)$,

$$(K_{A,0}U)(x) = \sum_{p=0}^{n-1}(1/p!)U^{(p)}(0)x^p + \sum_{p=n}^{\infty}(1/p!)[U^{(p)}(0) - A\cdot(M_{p-n}^{-1}(A(U^{(p)}(0))))]x^p$$

for all x in D. Hence,

$$\begin{aligned}(K_{A,0}U)(x) &= \sum_{p=0}^{n-1}(1/p!)U^{(p)}(0)x^p + \sum_{p=n}^{\infty}(1/p!)U^{(p)}(0)x^p \\ &\qquad - \sum_{p=n}^{\infty}(1/p!)U^{(p)}(0)(A\cdot(M_{p-n}^{-1}(Ax^p))) \\ &= U(x) - (Ax^n)\sum_{p-n}^{\infty}U^{(p)}(0)(A\cdot(M_{p-n}^{-1}x^{p-n}))\end{aligned}$$

since $Ax^p = (Ax^n)x^{p-n}$ for all x in D. Hence $(K_{A,0}U)(x) = U(x)$ if $Ax^n = 0$. Several examples are now considered in light of this observation.

EXAMPLE. 1. Take $m = 2$, $n = 1$. Suppose that $\begin{pmatrix} a \\ b \end{pmatrix}$ is a unit vector in E_2 and consider the simple partial differential equation $aV_1 + bV_2 = 0$. In the notation of this paper, this equation is written $Av' = 0$ where A is $\begin{pmatrix} a \\ b \end{pmatrix}$ or, more clearly, the linear functional dual to $\begin{pmatrix} a \\ b \end{pmatrix}$. If U is a real valued analytic function on the region D mentioned above, then in the notation of Section 4,

$$(K_{A,0}U)(x) = U(x) - (Ax)\sum_{p=1}^{\infty}(1/p!)U^{(p)}(0)\,(A\cdot M_{p-1}^{-1}x^{p-1}\ )$$

for all x in D where M_{p-1} is the transformation from $S_{2,p-1}$ to $S_{2,p-1}$ such that $M_{p-1}w = A(A\cdot w)$ for all w in $S_{2,p-1}$. Since $Ax = 0$ for all x in E_2 which are orthogonal to $\begin{pmatrix} a \\ b \end{pmatrix}$, one has that $(K_{A,0}U)(x) = U(x)$ for all such vectors x. Hence if $V = K_{A,0}U$, then V not only satisfies $aV_1 + bV_2 = 0$ but also satisfies the boundary condition $V(x) = U(x)$ for all x on the common part of L and D where L is the line through the origin which is orthogonal to $\begin{pmatrix} a \\ b \end{pmatrix}$.

Since L is orthogonal to the characteristic direction of the equation, it is, in a sense, an optimal line on which to specify boundary conditions. Note that $K_{A,0}U$ can, in this case, be extended by continuity to the set of all real valued functions on D which have at least one continuous derivative.

EXAMPLE 2. Suppose that $m=2$, $n=3$ and that each of A_1, A_2 and A_3 is in E_2 so that no two are linearly dependent. Suppose in addition that $A = A_1 \cdot A_2 \cdot A_3$, $\| A \| = 1$ and that D is a convex region containing the origin of E_2. Consider the problem of finding a member V of $A(D)$ so that $AV^{(3)} = 0$. If the dual of A_i is the point $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ of E_2, $i = 1,2,3$, then this equation can be written

$$C_1 V_{111} + C_2 V_{112} + C_3 V_{122} + C_4 V_{222} = 0$$

where $C_1 = a_1 a_2 a_3$, $C_2 = a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3$, $C_3 = a_1 b_2 b_3 + b_1 a_2 b_3 + b_1 b_2 a_3$, $C_4 = b_1 b_2 b_3$.

Clearly for this equation the three vectors $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$, $i = 1,2,3$, specify the characteristic directions. If U is in $A(D)$ and x is in D, then

$$(K_{A,0}U)(x) = U(x) - (Ax^3) \sum_{p=3}^{\infty} (1/p!) U^{(p)}(0) \, (A \cdot M_{p-3}^{-1} x^{p-3})).$$

Since $Ax^3 = (A_1 x)(A_2 x)(A_3 x)$ for all x in D, it is clear that $(K_{A,0}U)(x) = U(x)$ if x is in D and is orthogonal to one of A_1, A_2 or A_3. This indicates that $K_{A,0}U$ is a solution to a kind of "Goursat problem" $AV^{(3)} = 0$, $V(x) = U(x)$ if x is in D and orthogonal to one of A_1, A_2 or A_3. Hence $K_{A,0}U$ "ignores" all information contained in U except its values on lines through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ orthogonal to one of the characteristic directions. This observation seems to suggest a study of much more general third order partial differential equations on E_2 with three real characteristics and with data specified on lines orthogonal to the characteristic directions.

EXAMPLE 3. Suppose that $m = n = 2$, D is a convex region containing the origin of E_2 and A is chosen so that $A(x,y) = 2^{-1/2} xy$, the inner product of x and y, for all x, y in E_2. Then Laplace's equation can be written $A(U^{(2)}) = 0$.

Suppose that U is in $A(D)$ and denote $K_{A,0}\, U$ by Y. Suppose that t is a positive integer $\geqq 2$. Then, if x is in D,

$$Y^{(t)}(x) = \sum_{p=t}^{\infty} (1/(p-t)!) [U^{(p)}(0) - A \cdot (M_{p-2}^{-1}(A(U^{(p)}(0))))] x^{p-t}$$

and hence $Y^{(t)}(0) = (I - P_t) U^{(t)}(0)$ where $P_t w = A \cdot (M_{t-2}^{-1}(Aw))$ for all w in $S_{2,t}$.

Note that P_t is an orthogonal projection of $S_{2,t}$ onto a subspace $S'_{2,t}$ consisting of all vectors $A \cdot v$ for all v in $S_{2,t-2}$ and hence $I - P_t$ is the complementary projection. Denote by a, b an orthonormal basis of E_2 and denote by g and h the members of $S_{2,t}$ such that

$$g(a^{t-r} \cdot b^r) = \begin{cases} 1, r = 0,4,\ldots \\ -1, r = 2,6,\ldots \\ 0, r = 1,3,\ldots \end{cases} (r \leqq t) \text{ and } h(a^{t-r} \cdot b^r) = \begin{cases} 0, r = 0,2,\ldots \\ 1, r = 1,5,\ldots \\ -1, r = 3,7,\ldots \end{cases} (r \leqq t) \quad .$$

Some calculation gives that $gh = 0$ and $\| g \|^2 = \| h \|^2 = 2^{t-1}$ Denote $2^{-(t-1)/2} g$ by w and $2^{-(t-1)/2} h$ by z. Then $wz = 0, \| w \| = \| z \| = 1$. Since $S_{2,t}$ is of dimension $t+1$ and $S'_{2,t}$ is of dimension $t-1$, it follows that the pair w, z is an orthonormal set spanning the orthogonal complement of $S'_{2,t}$. Hence, if $Y^{(t)}(0) = (w(U^{(t)}(0)))w + (z(U^{(t)}(0)))z$, x is in D, and $x = ra + sb$, then

$$
\begin{aligned}
[Y^{(t)}(0)]x^t &= \sum_{p=0}^{t} \binom{t}{p} [(U^{(t)}(0))(a^{t-p} \cdot b^p)] [x^t(a^{t-p} \cdot b^p)] \\
&= (w(U^{(t)}(0))) \sum_{p=0}^{t} \binom{t}{p} [w(a^{t-p} \cdot b^p)][x^t(a^{t-p} \cdot b^p)] \\
&\qquad + (z(U^{(t)}(0))) \sum_{p=0}^{t} \binom{t}{p} [z(a^{t-p} \cdot b^p)][x^t(a^{t-p} \cdot b^p)] \\
&= 2^{-(t-1)} \mathrm{Re}\{[(D_a - iD_b)^t U](0)\} \mathrm{Re}[(r + is)^t] - 2^{-(t-1)} \mathrm{Im}\{[(D_a - iD_b)^t U](0)\} \\
&\qquad \mathrm{Im}[(r + is)^t] \\
&= 2^{-(t-1)} \mathrm{Re}\{\{[(D_a - iD_b)^t U](0)\} (r + is)^t\}
\end{aligned}
$$

where D_a, D_b denote directional differentiation in the a and b directions respectively. Knowledge of $[Y^{(t)}(0)]x^t$ for all x in D, $t = 0, 1, 2, \ldots$ gives, of course, a power series expression for Y.

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